Thus, the limits of applicability for problems of heat and mass transfer of the "chessboard" method are broader than follows from the stability conditions (9).

NOTATION

R_{in}, thermophysical transfer coefficients; R_{11}, coefficient of thermal diffusivity; R_{22}, diffusion coefficient; R_{12}, mass diffusion coefficient; R_{21}, thermal diffusion coefficient; v(x, t) heat function; u(x, t) mass function; h, τ, grid pitches; u_j^k, v_j^k grid analogs of the functions u, v; G, transition matrix; λ_m, eigenvalues of the matrix G; $F(\lambda)$, characteristic polynomial of the matrix G.

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SOLUTION OF THE TWO-DIMENSIONAL UNSTEADY DIFFUSION EQUATION FOR VORTEX FLOW

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A numerical method of solution based on the use of probability analogies is presented. An example of a calculation by the scheme developed is given.

Solid particles in fluidized bed devices take part in both random and directed motions in the form of circulating flows through the whole reactor [1, 2]. This circulation can be represented as a vortex superimposed on the diffusion intermixing of the solid phase. The intermixing process must then be described by an inhomogeneous differential equation for diffusion in vortex flow. It is very difficult or impossible to obtain an analytic solution of this equation. The method of finite differences is a universal method for obtaining approximate solutions of differential equations and is applicable to a broad class of problems [3].

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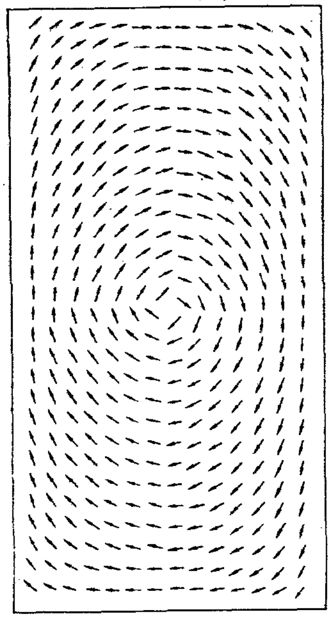

Fig. 1. Vortexflow pattern.

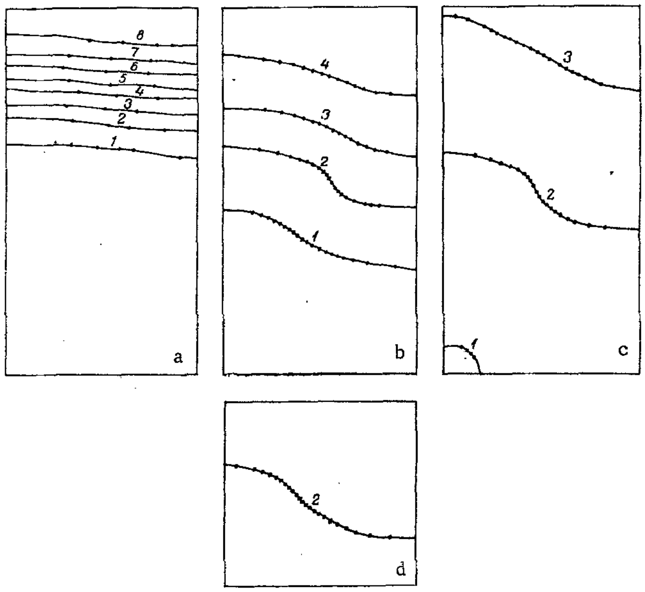

Fig. 2. Change of concentration distribution pattern with time. Numbers on isolines correspond to concentrations: a) $t=8.33 \cdot 10^{-4}$; b) $t=3.33 \cdot 10^{-3}$; c) $t=6.67 \cdot 10^{-3}$; d) $t=1.25 \cdot 10^{-2}$.

A necessary condition for the successful application of difference methods is the stability of the approximation scheme. The use of probability schemes guarantees stability. The Monte Carlo method was used in [4] to solve the two-dimensional steady diffusion equation, but the method of counting used becomes inefficient in determining the solution at all the mesh points.

We present a more economical method of solving the two-dimensional diffusion equation.

Suppose the equation

$$\frac{\partial c}{\partial t}=D\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right)+v_x\frac{\partial c}{\partial x}+v_y\frac{\partial c}{\partial y}. \tag{1}$$

is to be solved in a rectangular region $G=(0,a;\ 0,b)$ with a boundary F taking the boundary condition

$$\left.\frac{\partial c}{\partial n}\right|_F = 0 \tag{2}$$

and the initial condition

$$c(x,\ y,\ 0) = \frac{M}{a}\,\delta(y), \tag{3}$$

where v_x and v_y are components of the velocity $\vec{v}$ which satisfies the equations of continuity, vorticity perpendicular to the plane, and impenatrability of the boundaries:

$$\frac{\partial \vec{v}}{\partial x} + \frac{\partial \vec{v}}{\partial y} = 0, \tag{4}$$

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \Omega_z, \tag{5}$$

$$v_n|_F = 0. \tag{6}$$

Since (4) is satisfied, a vector potential $\vec{u}$ exists which for steady vortex flow satisfies Poisson's equation

$$\frac{\partial^2 \vec{u}}{\partial x^2} + \frac{\partial^2 \vec{u}}{\partial y^2} = -\Omega_z \vec{k} \tag{7}$$

and the boundary condition $\partial\vec{u}/\partial F = 0$, or in components:

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = -\Omega_z, \tag{8}$$

where $\Omega_z = \Gamma\delta(x-\xi,\ y-\eta)$, and Γ is the vortex strength. It follows from the boundary condition for u_z that $u_z|_F = \text{const}$. Since the velocity field is determined from u_z solely by differential operations, we can set $u_z|_F = 0$, or in more detail,

$$u_z|_{x=0} = u_{x=a} = 0,\ u_z|_{y=0} = u_z|_{y=b} = 0. \tag{9}$$

We solve the system (8), (9) by the method of separation of variables used by Grinberg to solve electrostatics problems [5]. The solution has the form

$$u_z = -\frac{\Gamma}{4\pi}\sum_{n=0}^{\infty} \lg\left[\frac{\left(\operatorname{ch}\frac{(2na+\xi-x)\pi}{b} - \cos\frac{\pi(y-\eta)}{b}\right)\left(\operatorname{ch}\frac{(2na+\xi+x)\pi}{b} - \cos\frac{\pi(y+\eta)}{b}\right)}{\left(\operatorname{ch}\frac{(2na+\xi-x)\pi}{b} - \cos\frac{\pi(y+\eta)}{b}\right)\left(\operatorname{ch}\frac{(2na+\xi+x)\pi}{b} - \cos\frac{\pi(y-\eta)}{b}\right)}\right] +$$

$$+\frac{\Gamma}{4\pi}\sum_{n=1}^{\infty} \lg\left[\frac{\left(\operatorname{ch}\frac{(2na-\xi-x)\pi}{b} - \cos\frac{\pi(y-\eta)}{b}\right)\left(\operatorname{ch}\frac{(2na-\xi+x)\pi}{b} - \cos\frac{\pi(y+\eta)}{b}\right)}{\left(\operatorname{ch}\frac{(2na-\xi-x)\pi}{b} - \cos\frac{\pi(y+\eta)}{b}\right)\left(\operatorname{ch}\frac{(2na-\xi+x)\pi}{b} - \cos\frac{\pi(y-\eta)}{b}\right)}\right]. \tag{10}$$

The components of $\vec{v}$ can be found by differentiating (10) with respect to x and y. After this we proceed to the solution of the system (1)-(3).

We write the difference scheme for (1)-(3):

$$\begin{gathered} c(x_i,\ y_j,\ t_{k+1}) = p_1(x_i,\ y_j)\,c(x_{i+1},\ y_j,\ t_k) + \\ + p_2(x_i,\ y_j)\,c(x_{i-1},\ y_j,\ t_k) + p_3(x_i,\ y_j)\,c(x_i,\ y_{j+1},\ t_k) + \\ + p_4(x_i,\ y_j)\,c(x_i,\ y_{j-1},\ t_k), \end{gathered} \tag{11}$$

$$\begin{gathered} c(0,\ y_j,\ t_k) = c(x_1,\ y_j,\ t_k), \\ c(a,\ y_j,\ t_k) = c(a-\Delta x,\ y_j,\ t_k), \\ c(x_i,\ 0,\ t_k) = c(x_i,\ y_1,\ t_k), \\ c(x_i,\ b,\ t_k) = c(x_i,\ b-\Delta y,\ t_k), \end{gathered} \tag{12}$$

$$c(x_i,\ y_i,\ 0) = \begin{cases} \dfrac{M}{a\Delta y}, & \text{if}\quad y_j = 0, \\ 0, & \text{if}\quad y_j \neq 0. \end{cases} \tag{13}$$

We require that

$$\sum_{s=1}^{4} p_s(x_i, y_j) = 1 \tag{14}$$

and

$$p_s(x_i, y_i) \geqslant 0. \tag{15}$$

The scheme (11) approximates Eq. (1) and satisfies condition (14) for the following values of the coefficients:

$$\begin{aligned}
p_1(x_i, y_j) &= \frac{(\Delta y)^2}{2[(\Delta x)^2 + (\Delta y)^2]}\left(1 + \frac{v_x(x_i, y_j)\,\Delta x}{2D}\right),\\
p_2(x_i, y_j) &= \frac{(\Delta y)^2}{2[(\Delta x)^2 + (\Delta y)^2]}\left(1 - \frac{v_x(x_i, y_j)\,\Delta x}{2D}\right),\\
p_3(x_i, y_j) &= \frac{(\Delta x)^2}{2[(\Delta x)^2 + (\Delta y)^2]}\left(1 + \frac{v_y(x_i, y_j)\,\Delta y}{2D}\right),\\
p_4(x_i, y_j) &= \frac{(\Delta x)^2}{2[(\Delta x)^2 + (\Delta y)^2]}\left(1 - \frac{v_y(x_i, y_j)\,\Delta y}{2D}\right).
\end{aligned} \tag{16}$$

To satisfy Eq. (15) we choose Δx and Δy so that

$$\left|\frac{v_x(x_i, y_j)\,\Delta x}{2D}\right| \leqslant 1 \text{ and } \left|\frac{v_y(x_i, y_j)\,\Delta y}{2D}\right| \leqslant 1,$$

with

$$\Delta t = \frac{(\Delta y)^2(\Delta x)^2}{2D[(\Delta x)^2 + (\Delta y)^2]}. \tag{17}$$

Equation (11) admits a probability interpretation; the coefficients $p_s(x_i, y_j)$ are determined analytically and can be interpreted as transition probabilities.

An important condition for the use of difference methods is the satisfactory accuracy of the solution. Since the scheme is stable with respect to the right-hand side [3], its accuracy is the same as the order of approximation and is $O((\Delta x)^2 + (\Delta y)^2)$. With this scheme the process can be calculated not only with boundary conditions of the first kind but also with boundary conditions of the second and third kinds, and also in the presence of vortices, since in this case the velocity at a point is the vector sum of the velocities induced at this point by each vortex separately.

The process can be investigated with an arbitrary initial concentration distribution and a different ratio of the sides of the apparatus. As an example a calculation was performed with the following data: $a = 1.5$, $b = 2.9$, $\xi = 0.75$, $\eta = 1.45$, $D = 150$, $\Gamma = 4$, $M = 10$, $\Delta x = \Delta y = 0.1$.

The time step was found from Eq. (17),

$$\Delta t = 1.67 \cdot 10^{-5}.$$

The vortex flow pattern is shown in Fig. 1. In the limiting case of $\vec{v} \equiv 0$ pure diffusion occurs. For $D = 0$ Eq. (1) degenerates into a linear differential equation.

In all intermediate cases when $0 < |vb/D| < \infty$ diffusion by "tongues" is superimposed on the vortex flow pattern.

Figure 2 shows isolines of the concentration field at various times. It is clear from the figure that the concentration levels out with time and the process approaches a steady state (Fig. 2d).

NOTATION

G, rectangular region; a, width of region G; b, length of region G; F, boundary of region G; t, time; x, y, linear coordinates; D, effective diffusion coefficient; v_x, v_y, velocity components; n, normal to boundary; c(x, y, t), concentration at point (x, y) at time t; M, mass of material; Ω_z, component of vorticity; $\vec{u}$, vector potential; Γ, vortex strength; ξ, η, coordinates of vortex; x_i, y_j, t_k, space–time coordinates of mesh points; Δx, Δy, Δt, coordinate and time steps.

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